Innovation Configuration for Mathematics

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**Overview of Multi-Tiered Instruction**

It is generally recognized that students require different types and intensities of instruction to master learning objectives. Multi-tiered systems of support (also called response to intervention) systematize the use of student performance data to distribute types and intensities of instruction and evaluate instructional effects at the district, school, classroom, and individual student level. MTSS is therefore a method of system improvement that seeks to improve learning outcomes for all students. In contemporary instructional systems, this notion that “one size does not fit all” when it comes to instruction is understood and has influenced policy recommendations in special education and content area instruction. It is not sufficient for teachers to present material to students and expect that all students will master the necessary skills. Instead, teachers should expect that ongoing assessment of student mastery of targeted skills serve as necessary feedback to the teacher about where additional and more intensive instruction is needed. Further, the teacher must be equipped to interpret student data to identify the type of instruction that will be most effective given the student’s level of proficiency and local expectations for learning outcomes.

The purpose of IC Math is to provide State Education Agencies (SEA), Institutes of Higher Education (IHE), and other stakeholders a tool to evaluate and provide a foundation for improving current state licensure requirements, state-, district-, and school-level professional development activities, and teacher preparation with respect to PK-12 mathematics, particularly for struggling learners including students with disabilities. The IC Math rubric is designed to address critical areas of knowledge and skills for mathematics teachers generally but also within an RTI/MTSS framework. The structure of the IC Math
rubric is based on several guiding principles - Teachers must understand and demonstrate mastery of the mathematics content they will teach; Teachers must understand how students learn mathematics content; Teachers must decide what content to teach; Teachers must decide how to teach that content; And teachers must know how to evaluate instructional effects and adjust forthcoming instruction to refine and enhance student learning.

**Teacher Mastery of Mathematics Content**

Ma’s (1999) seminal treatise on mathematics instruction in the U.S. is compelling evidence that teachers must have an understanding of the mathematics content they are expected to teach. Ma compared experienced and inexperienced mathematics teachers in the U.S. and China, asking teachers to first solve and then explain how they would teach students to solve certain mathematics problems. Despite Chinese teachers having less advanced training to become teachers (e.g., many had high school degrees only), the Chinese teachers handily outperformed U.S. teachers in generating correct problem solutions to the problems. Alarmingly, more than half of U.S. teacher participants could not solve the problem $1 \frac{3}{4}$ divided by $\frac{1}{2}$. Several key differences between U.S. teachers and Chinese teachers emerged in the Ma study. First, Chinese teachers were more competent in solving the mathematics problems in the study. Second, the Chinese teachers had a better sense of the key ideas and skills that underlie correct solution of a given problem. In other words, Chinese teachers were able to articulate the key prerequisite skills and understandings and explain to students how to apply past knowledge to solve the current problem. Third, Chinese teachers were fluent in demonstrating more than one way to solve each problem. Fourth, Chinese teachers were able to provide a mathematical proof to show
why a particular problem solution worked, whereas U.S. teachers tended to use (and encourage) trial and error which is both inefficient and does not facilitate consistently accurate understanding. Interestingly, U.S. teachers tended to teach a standard algorithm for solving a problem (e.g., “invert and multiply to solve division with fractions”); however, they were unable to explain why or how such an algorithm worked. The implication of an algorithm-only teaching approach is that once the algorithm is forgotten, students have no understanding of how or why such an algorithm worked so they are unlikely to be able to recreate it. Chinese teachers explain the algorithms--- they do not avoid them--- but they go a step further in demonstrating for students how and why multiplying a number by its reciprocal actually works with demonstrations using whole number division, converting fractions to division problems, such as $\frac{1}{2} = 1$ divided by 2 and demonstrating with whole number operations, using mathematical strategies like inverse operations, commutative law, and creating equivalence to solve for an unknown as multiple mathematical proofs of why the “invert and multiply” algorithm works. When students understand how the operation works, they can develop expectations for what a reasonable problem solution might be which decreases errors and deepens understanding.

Chinese teachers were found to spend significant time establishing mastery for pre-identified essential skills (such as addition and subtraction 0-20, rapid composition and decomposition of tens as a prerequisite to place value problem solving), to know in advance which ideas were the new key ideas to be established with instruction and to explicitly connect the new ideas to past ideas through mathematical proofs. When Chinese teachers selected tools to facilitate understanding, they were much more likely to correctly align tool selection with the key idea to be established and more often than not used
understanding of past operations to establish new understandings. The implication of Ma’s work is that Chinese teacher’s facility in problem solving and more advanced mastery of the mathematical content caused them to be more effective teachers, and such a case was made through descriptive data.

So, what is it that teachers must have expertise in with respect to K-12 mathematics content? Historically, this question has been difficult to answer in the United States due to the fact that states and localities have independently determined the content of their school curricula. And over the years experts in mathematics and mathematics education have not always agreed. Recently, however, a greater level of consensus has been reached, at least among experts in mathematics and mathematics education and among most states, leading to the Common Core State Standards (CCSS) in mathematics and we will discuss these standards in greater detail in a section below. For now, these standards include eleven content domains or big ideas:

1. Counting & Cardinality
2. Operations & Algebraic Thinking
3. Number & Operations in Base Ten
4. Number & Operations – Fractions
5. Measurement & Data
6. Geometry
7. Ratios & Proportional Relationships
8. The Number System
9. Expressions & Equations
10. Functions
11. Statistics & Probability

As illustrated in Ma's study, there is more to the “content” of mathematics than the mathematical concepts and skills - the “what” of mathematics. The content of mathematics also includes the “doing” of mathematics, those mathematical practices that promote understanding, reasoning, problem solving, making connections, representing, and communicating mathematics in sophisticated ways. Therefore, the CCSS for mathematics emphasizes eight mathematical practices that are critical for developing the understanding and skill exemplified by the Chinese teachers in Ma’s study. The eight practices were distilled from the National Council for Teachers of Mathematics (NCTM) process standards (problem solving, reasoning and proof, communication, connections, and representation) and the National Research Council’s report *Adding It Up* (adaptive reasoning, strategic competence, conceptual understanding, procedural fluency, and productive disposition). The CCSS’s eight mathematical practices are:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.
However, expertise in mathematics content alone has not been shown to be sufficient for establishing more successful mathematical learning for students. Hattie (2009) found that subject matter knowledge was negligibly related to student achievement ($d = .09$). Ma, too, acknowledged the necessary but insufficient role of teacher knowledge, stating, “A teacher’s subject matter knowledge may not automatically produce promising teaching methods or new teaching conceptions...” but “…without solid support from subject matter knowledge, promising methods or new teaching conceptions cannot be successfully realized.” (p. 38, Ma, 1999).

Slavin and Lake (2008) conducted a synthesis of experimental and quasi-experimental research on mathematics achievement and applied rigorous criteria to include studies of sufficient quality to permit meaningful conclusions (87 of 256 studies reviewed met their inclusion criteria). There were three general types of studies: Studies of mathematics curricula (n = 13), studies of computer-based instruction (n = 36) and studies designed to change the interaction between student and teacher during mathematics instruction (e.g., increase teacher feedback) (n = 36). In general, studies designed to change teacher-student interaction were of the highest experimental quality among included studies. Curricula effects were weak ($d = .10$). Computer-based instruction generally involved practicing mathematics facts on a computer and produced a small to moderate average effect ($d = .19$). However, studies that assessed and changed teacher-student interaction during mathematics produced a moderate achievement effect that was on average three times the effect of the mathematics curricula ($d = .33$). The most powerful lesson of the research data is that content knowledge (knowing what to teach) and the science of effective instruction
(knowing how to teach) are both essential prerequisites to successful learning outcomes for students in mathematics.

Necessary indicators that teachers have mastered the mathematics content they will be teaching include:

1. Demonstrates understanding of mathematics content for grade range they are/will be certified to teach (including content for grade levels that are immediately previous to and subsequent to this grade range).

2. Demonstrates abilities to provide mathematical proof and reasoning for solutions to problems.

3. Demonstrates understandings of how mathematical concepts and skills within and across domains (i.e., Counting and Cardinality, Operations & Algebraic Thinking, and so on) interrelate and build upon one another over time.

4. Demonstrates ability to model multiple processes for engaging students in doing mathematics (multiple methods of solving problems).

In this section, we have indicated the need for teachers to know the mathematics content they will teach. In the next section, we will discuss the necessity of a teacher knowing how to teach mathematics.

**Teacher Mastery of Student Learning**

Instruction is a science with measurable implementations and measurable effects on learning. Instruction can be high-quality or low-quality and certain hallmark indicators signal high-quality (or effective) instruction. One hallmark of effective instruction is the use of frequent assessment (Fuchs, Deno & Mirkin, 1984). In one study, Yeh (2007) found that use of frequent assessment was four times as effective in producing measurable
achievement gains as a 10% increase in per pupil spending. Assessment in mathematics may be formal and standard as in the case of using screening data to evaluate overall mathematics instructional adequacy at the district, school, and classroom level and in selecting small groups or individual students in need of intervention or support to attain expected competencies. Assessment may also be informal as in the case of frequent checking for student understanding when teaching a new concept or skill. The use of frequent assessment underscores the need for effective instruction to take into account the learner competencies before instruction begins and as it progresses. Student assessment is a necessary feedback loop to the teacher telling the teacher whether the instruction has succeeded or not.

To accomplish data-driven instruction, teachers must have a strong understanding of the content to be taught and learned, student assessment data to indicate whether students are mastering the concepts and skills that are being taught, and a system for responding to children who are not experiencing success. Teachers must use their knowledge about how students typically learn certain mathematics skills and concepts, common barriers to learning mathematics, and must have sufficient facility with the mathematics content itself to adjust instruction for a better result. Teachers cannot simply present the content that appears in the adopted textbook and presume that learning will follow. Instead, teachers must provide sufficient acquisition instruction to establish understanding for students given the students’ current level of mathematical understanding and mastery of associated or prerequisite skills. Aligning instruction with student performance means that teachers will provide mathematical explanations that rely on concepts and skills the student has already mastered. When the teacher expects children
to approach mathematics in the same way that adults do, the teacher may fail to recognize that a student’s mathematical thinking is accurate (e.g., when the student uses an alternative algorithm or problem-solving strategy) (Carpenter, et al., 1999) or the teacher may encourage and reward the use of a procedure or process that is mathematically accurate but may not be appropriate for a particular student because of the student’s age or lack of prior mathematical knowledge. For example, students typically build multiplicative reasoning using addition (i.e., repeated addition) before they begin using multiplicative strategies like partial products or the distributive property. When transitioning from reasoning with addition to reasoning with multiplication, students begin to utilize skip counting and use of arrays. A teacher who does not recognize where students are in this developmental sequence may expect students to engage in mathematical reasoning for which they are not ready, for example expecting a student who has not yet developed multiplicative reasoning to utilize partial products to solve a 2 by 2 digit multiplication equation, which will likely lead to frustration and lack of success for that student. When teachers are able to combine their knowledge of how students typically approach mathematics at particular grades/ages/developmental levels with recognition of the particular prerequisite knowledge and skills a student lacks via appropriate assessment data, they can adjust and focus instruction on the mathematics that a student needs most in ways that connect to how the student currently thinks and learns mathematically.

Instructional practices that are aligned well with student need will produce stronger learning gains (Burns, Codding, Boice, & Lukito, 2010). Specifically, when students are learning a new skill, student performance will likely be inaccurate or incomplete, and effective instructional practices will include modeling of correct and incorrect responding,
use of prompts and cues to establish correct understandings, providing elaborate and immediate corrective feedback to advance student understanding, providing a repetition loop such that the student has an immediate opportunity to correct errors with support from the teacher. When a student is acquiring a new skill, the teacher must understand and make explicit how the new skill is connected to existing knowledge and be able to demonstrate why a particular response is correct (using mathematical proof with operations the student already knows how to do) and why a different response is incorrect. The teacher must be able to anticipate common misunderstanding and errors that tend to occur when teaching new skills and both precorrect for these errors up-front and monitor for their occurrence as instruction progresses. Assessing to verify that students have mastered the relevant prerequisite skills should occur and assessment to verify student understanding should also occur during acquisition instruction.

Students with more significant mathematics difficulties (e.g., due to learning disabilities) can present slightly different challenges for teachers during acquisition instruction. Poor inhibition control occurs when students with learning disabilities have difficulty filtering out irrelevant mathematical associations and students with learning disabilities may be particularly susceptible to a lack of inhibition control (Geary, et al., 2000; Geary et al., 2007). For example, when recalling basic addition facts such 4 + 2, students with learning disabilities may associate the individual digits 4 and 2 with the number that follows them in a counting sequence, in this case 5 and 3. Therefore they respond with the sum 8 instead of 6. In a study by Geary et al. (2000) students identified with mathematics disabilities only, combined mathematics and reading disabilities, and reading disabilities only made more addition fact retrieval errors compared to their
typically achieving peers. The pattern described above was a common pattern among the students with learning disabilities. Students with learning disabilities can also have difficulties with visual-spatial processing and associating mathematical language with the abstract notations (i.e., numbers and symbols) used to represent mathematics. Explicit instruction in mathematics vocabulary and symbols may be necessary to establish accurate and fluent performance. Providing students with antecedent cues to prompt the correct operation (e.g., highlighting the operation in the problem) and immediate corrective feedback when the wrong operation is performed are powerful strategies that can be used to correct this problem. Students with both mathematics disabilities and reading disabilities are especially at risk of mathematics difficulties (Jordan, 2007) because they have trouble with both the language and the number sense demands placed on them when learning and doing mathematics affecting students abilities to problem solve whether it occurs within the context of word problems or not. Other student learning characteristics that can impact mathematics include metacognitive deficits, learned helplessness, passive approaches to learning, cultural and linguistic diversity, mathematics anxiety, and academic skill gaps (Allsopp, Kyger, & Lovin, 2007; Miller & Mercer, 1997). Teachers must be able to anticipate, identify, and troubleshoot the difficulties that some students will experience that will interfere with learning. For some students, the solutions will be more complex and intensive than for other students. The complexity of the intervention should be guided by the student’s need regardless of whether the student is eligible for special education services or not. For students who require more intense instruction to master what is being taught, schema-based instruction (Jitendra et al., in press) and explicit
teaching for transfer instruction (Fuchs et al., 2003) are models that can work when properly used.

All mathematics content is not equivalent. Some content is easier than other content to understand for most people. Teachers must understand and anticipate challenges that are associated with certain mathematical concepts. Krasa and Shunkwiler (2009) discuss this with respect to the ability to decode (read) and encode (write) numerals and mathematical symbols. Visual-spatial skills and how they integrate with memory are important because numerals and mathematical symbols have distinct shapes and orientations in space. Students must be able to discriminate fine differences between numerals and between other mathematics symbols and notations using iconic visual memory skills. The meaning of mathematic symbols is often predicated upon its orientation in space (e.g., the inequality signs > and <). The extent to which students are able to both reliably store in and retrieve from memory the images of numerals and symbols accurately will dictate students’ abilities to efficiently decode and encode mathematic numerals and symbols. Instruction to establish fluent responding to mathematical symbols is important with some students requiring more trials to mastery than others.

Fractions are a particular area of difficulty for most U.S. students. Krasa and Shunkwiler (2009) say that conventional fractions turn the meaning of number “upside down.” Fractions are the first instructional occasion in mathematics for which the base unit is not 1 (or cannot be made to be 1) which causes numbers and operations to function differently than students have seen up to this point in their mathematics instruction. For example, the numbers used in fractions do not have a natural counting sequence and they
do not have a unique number representation for each fraction. So a young student understands that $4 = 4$ and no other number by itself can be made to equal 4. With fractions, two values (e.g., $1/4$ and $2/8$) are equivalent numerical quantities but they use different numbers. The relationship between magnitude and number is also an issue with fractions. Consider the fractions $1/3$ and $1/7$. The fraction with the “bigger” number in the denominator is smaller than the fraction with the “smaller” number, which turns the student’s understanding of cardinality and ordinal position upside down. When multiplying whole numbers, the product is always a greater number and when dividing whole numbers, the quotient is always a lesser number. However, the opposite is true when multiplying and dividing fractions. Fractions are one example of content that presents challenges to students that teachers can anticipate and plan for in advance of instruction. Knowing that fractions cause confusion to most students when they are introduced is information that can be used by teachers to better explain how fractions work, how they relate to whole numbers and whole number operations, and how they expand a student’s capacity for solving mathematical problems. Regrouping, multi-digit multiplication, and division with remainders offer predictable opportunities for confusion and error that teachers can anticipate, take action to prevent misunderstanding and student frustration, and use the occasion to expand the student’s mathematical proficiency.

Once a skill has been established, children should be provided with instruction that is designed to build fluency. Fluent performance represents more advanced skill mastery, forecasts retention of the learned skill over time, and forecasts the ability to apply or adapt the skill to solve novel and more complex problems. Two children may score 100% accurately on a task, but one may be much less proficient than the other (and need different
instruction than the other). Imagine a child who can accurately provide an answer but has to draw and count hash marks, double check the answer, and provides a halting and incomplete explanation for the solution. Now, imagine a child who answers the problem immediately, without hesitation, and when asked to explain it can explain the solution and possibly even solve the problem a different way. The second student is more proficient and fluency will reflect this. Fluency moves beyond accuracy of responding and adds a timed dimension to the response (Johnson & Layng, 1992) and has been defined as accuracy plus speed (Binder, 1996). In mathematics, computational and procedural fluency can be measured with digits correct per unit of time on a content-controlled task. There is strong consensus that conceptual understanding and fluent skill performance are intertwined and bidirectional such that one begets the other and vice versa and excellent instruction emphasizes both (NMP, 2008). In 2001, the NRC (precursor to the NMP) stated, “Procedural fluency and conceptual understanding are often seen as competing for attention in school mathematics. But pitting skill against understanding creates a false dichotomy. As we noted earlier, the two are interwoven. Understanding makes learning skills easier, less susceptible to common errors, and less prone to forgetting. By the same token, a certain level of skill is required to learn many mathematical concepts with understanding, and using procedures can help strengthen and develop that understanding.” (p. 122, NRC, 2001).

Performing a skill fluently reduces the difficulty associated with solving multi-step problems, application problems, and learning new related content in the future. Yet, fluency-building instruction is often overlooked or deemphasized in mathematics instruction at a substantial cost to student achievement in the U.S. (Loveless, 2003).
Fluency-building instruction is appropriate for students who can accurately respond without adult support and includes strategies like assessing/monitoring fluent skill performance, setting goals for more fluent performance over time, and frequent, uninterrupted practice intervals with delayed corrective feedback. Because fluency-building instruction commonly receives inadequate focus or is ineffectively carried out, it is a ripe target for mathematics performance improvement in many schools. Fluency-building instruction should occur daily and should target a skill that students know how to perform accurately. Many readers may think of “drill and kill” worksheets as representing fluency-building instruction and this association is not accurate in most places. The way most teachers use worksheet practice is not optimally effective because (1) the skill is not selected based upon student performance, (2) skill difficulty and content is not advanced based on student performance gains, (3) corrective feedback is not provided immediately following the practice interval, (4) the problems the child is exposed to are always presented in the same order using the same format, (5) goals and rewards are not provided for more fluent performance and performance is not tracked at all to show growth. When fluency-building strategies are aligned with student proficiency and provided in a high-quality way, achievement gains are observed (Fuchs, Fuchs, Mathes, & Simmons, 1997; VanDerHeyden & Burns, 2005; VanDerHeyden, McLaughlin, Algina, & Snyder, 2012).

For children who struggle to master essential grade-level mathematics skills, daily fluency-building instruction on prerequisite skills at the student’s instructional level is a powerful method to close achievement and performance gaps. For students who struggle, small-group supplemental instruction is ideal for fluency-building instruction on prerequisite and foundational skills as a powerful complement to core instruction. Students
with disabilities who persistently struggle may require many more trials to reach proficiency and these trials can be provided with multi-tiered instruction.

Once children can respond independently and fluently, then opportunities to generalize the skill should be provided. Generalization problems can include conversion of problems to easier problems; creating equivalent problem solutions; solving for unknowns; and solving word problems. Students need multiple opportunities to apply concepts and skills with which they have become fluent within problem solving situations. Problem solving has to do with much more than solving word problems. It involves finding solutions to applied problems that are situated in different contexts, ranging from tangible to abstract. Applied problem solving requires students to utilize both conceptual and procedural understandings within or across multiple mathematical domains or big ideas (e.g., solving for an unknown to solve an angle, applying a known rate to solve for an unknown unit of time). Problem solving also requires students to apply strategies that organize their thinking about the problem and how to solve it. When students are able to generalize individual skill sets and even adapt their mathematical understandings to build new understandings, they become effective mathematical problem solvers. In contrast to students learning isolated sets of mathematics concepts and skills for a particular grade level or course, this type of mathematical thinking and skill is the cornerstone of the CCSS – Mathematics. The foundation for this mathematical competency is built by providing all children with supported opportunities to solve problems and discuss and justify problem solutions every day. In many mathematics homework assignments, there are a great number of computation problems and only one to two “thinking” problems, which does not provide adequate practice opportunities for most students. Teachers can and should
provide sufficient practice solving applied problems and asking students to “think out loud” or “teach someone else” how you solved the problem, beginning at Kindergarten and requiring gradually more sophisticated explanations as learning progresses.

In conclusion, using student assessment to align instructional practice with student need is one hallmark of effective instruction. Additionally, emphasizing multiple student opportunities to respond, maximizing corrective feedback (frequency, quality, immediacy) for students learning new skills, encouraging and maintaining high motivation for students, and providing guided opportunities to solve problems and communicate, explain, justify, and compare problem solutions are hallmarks of effective instruction. Teachers must understand how students typically learn mathematics, identify common barriers to mathematical understanding, and adjust instruction to prevent misunderstanding and facilitate proficiency.

Necessary indicators that teachers have mastered an understanding of how students learn and how to facilitate that learning include:

1. Demonstrates an understanding of how typical students’ mathematical thinking develops over time for foundational concepts.

2. Demonstrates understandings of common mathematical misconceptions and error patterns that represent faulty mathematical thinking.

3. Demonstrates understandings of student performance characteristics that forecast successful future learning in mathematics (i.e., has a system to identify students who require support to gain mathematical proficiencies and a system for knowing what type of instructional support is needed).
In the next section, we will detail the importance of a teacher deciding what skills and concepts to teach.

**Planning Instruction: Deciding What to Teach**

In mathematics there has been a trend over the last two decades to streamline mathematics education and address the critique that mathematics instruction in the U.S. has been a mile wide and an inch deep. Toward that end, policy groups have made recommendations for identifying a honed-down set of essential skills for which sufficient mastery would create more mathematically competent students. The first outcome of this movement in mathematics was the Principles and Standards for School Mathematics (NCTM, 2000) and the Curriculum Focal Points documents published by the National Council of Teacher’s of Mathematics (NCTM, 2006) which appears to have been highly influential to the National Mathematics Advisory Panel (2008) and most recently the Common Core State Standards in mathematics (2010, available at http://www.corestandards.org/Math). The curriculum focal points provided a streamlined guide to essential learning outcomes in number and operations, data analysis, measurement, and algebra, defined the essential skills specific to each grade level, and explained how grade-level skills were connected to skills learned at earlier grade levels.

The National Math Panel (NMP) report delineated a smaller subset of three foundational concepts/skills for algebra readiness specifically including fluency with whole numbers, fluency with fractions, and particular areas of geometry and measurement (e.g., perimeter and area of triangles and quadrilaterals, properties of two and three dimensional shapes, relationships between triangles and slopes of lines). The NCTM included focal points specific to algebra but did not specify a set of skills that are necessary for success in high
school Algebra. However, a comparison of the NMP and the FP show that there are more similarities in terms of grade-level expectations than there are differences. The NMP suggested that students be proficient in whole number addition and subtraction by the end of grade 3, which is one year later than the Focal Points (FP) document suggested. The NMP suggested proficient multiplication and division of whole numbers by the end of grade 5, which was consistent with the FP. The NMP and FP documents were consistent in recommending: representing and comparing fractions and decimals on a number line by end of Grade 4, comparing fractions, decimals, and percent by the end of grade 5, adding and subtracting fractions and decimals by grade 5, multiplying and dividing fractions and decimals by grade 6, and solving percent, ratio, and rate problems and extending work to proportionality by end of grade 7. The NMP and FP included mastery of operations with positive and negative fractions and integers. There was consistency in essential skills identified in geometry and measurement between the NMP and FP, however, the NMP required mastery of solving perimeter and area of triangles and quadrilaterals one year later than suggested by Focal Points (grade 5 for NMP, grade 4 for FP), and the same was true for analyzing properties of two-dimensional shapes to solve for perimeter and area and three-dimensional shapes to solve for surface area and volume (grade 6 for NMP, grade 5 for FP).

The Common Core State Standards (CCSS) also were apparently influenced by earlier work to streamline learning expectations in mathematics. The CCSS emphasized mastery of number by grade 3 including operations, relationships between operations, and place value understandings. CCSS emphasized understanding of number and operations related to fractions by Grade 4 and understanding of decimals and the rate of
decomposition in moving from left to right (or composition in moving from right to left) by Grade 5. Specifically, the CCSS suggested mastery of skills including fluent addition and subtraction of whole numbers within 20 by grade 2, fluent addition and subtraction of whole numbers within 100 by grade 3, fluent multiplication and division of whole numbers within 100 by grade 3, mastery of the relationship between operations of whole numbers by Grade 3 (e.g., ability to convert multiplication problems to addition, to identify the inverse operation and solve for an unknown, to convert more challenging problems to easier problems using the relationship of operations), multi-digit multiplication and division by grade 4 with mathematical explanations, operations with decimals by grade 5, operations with fractions by grade 5, and use of ratios, proportions, operations with fractions, factors, multiples and negative numbers to solve problems by grade 6. In summary, there is fairly consistent overlap between the three most recent and most influential policy documents offering guidance to classroom teachers about which skills should be established by what time points during a child’s learning career in mathematics, with the CCSS expecting mastery of skills slightly earlier than the NMP and FP documents (e.g., whole number multiplication and division by grade 3 in CCSS compared to grade 5 in NMP). The CCSS are the most detailed set of learning standards in mathematics that have benefitted from the efforts of NCTM’s FP and the NMP and offer teachers a sound basis for planning instruction and verifying student mastery of essential grade-level concepts and skills.

A specific sequence of concepts and skills provides a map for teachers to follow to guide students to mathematical understanding. Because instruction will unfold across many years, it is important for all teachers to take a multi-year view of math learning and
instruction. The sequence of skills provides teachers with an identified set of outcomes of their instruction and an understanding of how prerequisite skills relate to current instruction.

Ma referred to the notion of skill sequences as “conceptual maps” and found that knowledgeable teachers understood what the most important prerequisite skills were for new skills and could specify the “big idea” that was being taught when a new skill was introduced. The expected learning outcomes, in sequence, give the teacher a set of skills to assess to determine if instruction is working as desired and to identify which students might need additional support. The standards also provide schools and districts with a system for knowing how well students are mastering the mathematics content that has been identified as most critical to long-term school and career success.

When a teacher is introducing a new skill to be learned, the teacher should be clear about which skills and understandings must precede understanding of the new concept, determine whether students have mastered the prerequisite skills and concepts, introduce the new skill using existing knowledge to explain how it works, and finally, understand what new concepts and skills can be built upon the new knowledge in future instruction. As noted earlier, fluency in prerequisite skills forecasts successful learning of related future skills, especially when the new skill is introduced using high-quality acquisition instruction strategies. It is not enough to simply provide the rule, model the solution, and encourage students to memorize the steps in the procedure or algorithm. A teacher must be able to model multiple ways to solve a problem, explain and mathematically demonstrate how one solution works when another does not, and then directly and explicitly teach the algorithm. Conceptual understanding fluency is the ability to solve a problem in multiple ways, to
explain why a given solution works, to explain how the concept is connected to other mathematical concepts and skills, and to estimate correct problem solutions for related problems. Conceptual fluency is distinct from a halting, incomplete explanation or one that simply repeats the algorithm, because algorithms that are not understood will be forgotten and the child will have no way of reasoning his or her way to a correct problem solution which is at odds with the very logical and coherent natural structure of mathematics. When children understand the “big ideas” they do not have to rely on memorized tricks to solve problems. The CCSS has stated, “Asking a student to understand something means asking a teacher to assess whether the child has understood it.”

Establishing conceptual understanding does not mean that children should be taught using only easy and easy-to-visualize problems. Establishing conceptual understanding means that teachers use mathematical proofs to demonstrate why an error is an error (i.e., how it interferes with correct problem solution) and what other logical strategies can be applied to find the correct problem solution using what students already know. Wu masterfully explains the fallacy of shying away from algorithms in the name of advancing conceptual understanding stating, “... the resistance that some math educators (and therefore teachers) have to explicitly teaching children the standard algorithms may arise from not knowing the coherent structure that underlies these algorithms: the essence of all four standard algorithms is the reduction of any whole number computation to the computation of single-digit numbers.” Hung Hsi-Wu, p. 9 American Educator (1999). Wu’s point is that it is a misguided effort to teach students to solve only problems they can draw or visualize and then encourage trial-and-error learning as the basis for conceptual understanding. Instead, teachers should use single-digit operations and student’s
understanding of these computations to explain how more complicated problems can be solved. In Ma's study, for example, one highly competent Chinese teacher demonstrates why subtracting a 3-digit number from a 3-digit number from right to left is not necessary but is ultimately more efficient when regrouping is required. This demonstration involves solving a subtraction problem from left to right and then having to back-up to decompose a higher-value unit and erase and change the digit that was already written in the solution space.

Similarly, when teaching the process for multi-digit multiplication, a highly competent teacher may show that it is not necessary to multiply from right to left so long as the place value properties are maintained in the products that will be added together for the final solution. Working from left to right is inefficient, however, and causes the problem solver
to have to back up and change solutions if regrouping (e.g., composing a higher-value unit) is required.

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Teaching multi-digit multiplication in this way makes the place value properties that undergird the algorithmic solution explicit, which makes the algorithm sensible to the student, and one that can be more easily remembered or re-created if forgotten.

Deciding what skills to teach requires teachers to begin with the standards and develop a plan for ensuring prerequisite skills are intact, the new skill is established, and verifying children are ready for the new concepts that will follow. Examples are provided for five skills in the Table 1.

<Insert Table 1 about here>
Schoolwide screening in mathematics should focus on skills identified in the CCSS for a particular grade level that students are expected to master in order for the students to benefit from grade-level instruction. This is critical because targeting grade level knowledge and skills will support successful future learning in mathematics. Student data can be examined at the school, grade, and class level to determine the extent to which most students are mastering the learning outcomes that are expected. Where large numbers of students have not mastered expected skills, the adequacy of core instruction should be questioned and corrective efforts should be designed to improve the efficacy of core instruction. Corrective efforts should include repairing widespread prerequisite skill deficits, ensuring high-quality core instructional practices, and increasing the rate of progress monitoring and instructional adjustments at the class, grade, school, and district level. In classes where the majority of students are performing above the risk criterion and demonstrating mastery of expected skills and concepts, then individual students may be identified as needing additional support. These students may be students who scored in the nonproficient range on the preceding year-end test or they may be students who have not demonstrated mastery of a concept or skill when the majority of the student’s classmates have done so and it’s time for instruction to move on. For these students, a small-group supplemental intervention or intensive individualized intervention may be offered to ensure the student reaches mastery.

For students in need of intensive instruction, the foundational big ideas that structure the NCTM’s FP, the Algebra Readiness standards of the NMP, and the CCSS can provide guidance for what concepts/skills should be the focus of assessment and instruction. Although each set of big ideas/standards differ slightly in scope and purpose,
there is more overlap than difference in the concepts/skills identified as most important outcomes of instruction at given grade levels. Thus, the learning standards, and especially the most recent set of CCSS in mathematics, are invaluable resources for designing intensive mathematics interventions. For example, if the goal of a 5th and 6th grade intervention were to prepare students for success in Algebra 1 and Algebra 2, then utilizing the Algebra Readiness skill recommendations from the NMP to mastery of foundational skills and develop remedial interventions would be useful. For younger students, the FP can provide the problem solving team and intervention specialist with the foundational concepts/skills relevant to important content domains. Number sense development is an example because of its importance to later mathematical success (Griffin, Case, & Siegler, 1994; Griffin, 2004). The FP or the CCSS surrounding number and operations and algebra could be used to identify content-related targets for assessment and intervention for these students. Struggling learners must be engaged in doing mathematics in meaningful ways that will allow them to develop critical thinking and problem solving skills. The eight essential mathematical practices suggested by the CCSS (identified in an earlier section) provide interventionists a structure for different ways they can and should engage struggling learners in the content that is the focus of the intervention.

Necessary indicators that teachers can decide what skills to teach include:

1. Specifies the sequence of expected mathematics learning outcomes and places these learning outcomes on an instructional timeline with explicit consideration of multi-year learning goals.
2. Uses screening assessment to determine whether systemic learning deficits exist (classwide, gradewide, and course-specific). Uses screening data to identify students in need of supplemental support.

3. Emphasizes critical areas of mathematics foundational to mathematics success by targeting several big ideas per grade level or course for in-depth emphasis and continuous progress monitoring across the school year.

4. Uses assessment of just-taught skills to verify mastery of expected learning outcomes on time and to identify and diagnose student misunderstandings before advancing to new content. Uses assessment of just-taught skills to identify students for supplemental instruction. For children requiring intensive support, assessment should be more involved verifying mastery of prerequisite skills, conceptual understanding, and verifying the effect of particular intervention supports before they are deployed in the classroom.

5. Selects the learner objectives of each instructional lesson based on student performance data from previous lessons and the appropriateness of “fit” to previously mastered concepts, learner behaviors, and future skills to be learned. For children requiring supplemental support, aligning instruction to student proficiency may require working on prerequisite skills. For students requiring intensive support, alignment may require working on foundational mathematics concepts and skills.

6. Works collaboratively with other school personnel to appropriately integrate core instruction supports for students and teacher participates on decision team to verify
that all students show upward growth or learning gains including those students receiving supplemental and intensive supports.

In the next section, we will discuss the evidence-based indicators of effective instruction in mathematics.

**Planning Instruction: Deciding How to Teach**

Teachers often approach teaching mathematics from a particular theoretical or philosophical perspective. Teacher educators and researchers in mathematics can also be overly influenced by their particular theoretical or philosophical orientation toward mathematics and how it should be taught and learned (Parmer & Cawley, 1997). Preparing teachers to be effective mathematics educators for struggling learners is complex. The NCTM, within its Standards for Teaching Mathematics, includes six essential standards for professional development of teachers of mathematics that teacher preparation programs should incorporate: modeling good mathematics teaching, knowledge of mathematics, knowing students as learners of mathematics, knowing mathematical pedagogy, developing as teachers of mathematics, and the teacher's role in professional development (NCTM, 2000). Selecting instructional strategies from among those that have been shown to work in research (i.e., evidence-based) and ensuring the strategies are well-aligned with goals of instruction and the proficiency of students, monitoring strategy use and student learning outcomes, and making adjustments to instruction when learning is not occurring should be the guiding tenets in deciding how to teach.

When teaching new skills, teachers must make the connection to past understandings explicit, show how and why the algorithms work, provide detailed corrective feedback, and avoid the temptation to teach students to use trial-and-error as a
problem-solving strategy. Relying on the textbook alone to guide instruction is not sufficient (Slavin & Lake, 2008) and instead teachers should follow the learning standards provided by the CCSS, consider student data to verify that students have mastered prerequisite skills, introduce new ideas in ways that ensure conceptual understanding as we have defined it in this paper, make salient for the student how the new skill can be used to solve problems accurately, build fluency for skills that the student can independently and accurately perform, and provide opportunities to generalize (e.g., creating equivalent quantities, solving for an unknown, “undoing” an operation).

All students benefit from instruction that is systematic in nature, is clear and logically sequenced, provides sufficient opportunities to practice new skills, and examines data to evaluate and verify that students have developed conceptual understandings and can apply those understandings fluently in meaningful ways. It is likely that struggling learners, in particular, will need more explicit mathematics instruction characterized by more salient cues and prompts for correct responding, more sophisticated prompt fading techniques, a greater density of opportunities to respond over a longer period of time, more immediate and more elaborate corrective feedback, more frequent progress monitoring, more gradual increases in task difficulty, and guided practice solving related applied problems. The current research base, although limited, provides a beginning foundation for the kinds of instructional practices in mathematics that can positively affect struggling learners. For example, a promising Tier 2 instructional practice for struggling learners in mathematics is the utilization of a concrete-representational-abstract (CRA) sequence of instruction (Bryant, Bryant, Gersten, Scammacca, & Chavez, 2008; Flores, 2003; Fuchs, Compton, Fuchs, Paulson, Bryant, & Hamlett, 2005; Maccini & Hughes, 2000;
Witzel, 2005). A synthesis of research by Gersten, Beckman, Clarke, Foegen, Marsh, Star, and Witzel (2009) provides evidence-based recommendations for tier 2 and tier 3 practices at the elementary and middle school levels. Math instruction that is explicit and systematic and that includes models of efficient problem solving, verbalization of thought processes, guided practice, corrective feedback, and cumulative review is one practice. A second practice recommended is instruction in the structure of word problem types and how to discriminate superficial from substantive information for determining when to utilize problem solving strategies that students have already learned. The use of visual representations of mathematical ideas (e.g., manipulatives, drawings, graphs, number lines) and short approximately 10-minute daily sessions that attend to arithmetic fact retrieval are two other effective tier 2 and above practices recommended by the authors. Another promising mathematics instructional practice that has potential for success with struggling learners within MTSS includes anchoring problem solving in authentic and relevant contexts (Bottge, Heinrichs, Chan, & Serlin, 2001; Bottge, Heinrichs, Mehta, & Hung, 2002; Bottge, Heinrichs, Mehta, Rueda, Hung, & Danneker, 2004; Bottge, Rueda, & Skivington, 2006).

Necessary indicators that a teacher knows how to teach mathematics include:

1. Designs instruction to prevent misconceptions and errors and to rapidly detect and re-teach where misunderstandings occur. Highly explicit acquisition instruction may be necessary for students requiring intensive supports.

2. Designs instruction to facilitate students’ development of connections and understandings of relationships among mathematical concepts. For children
requiring intensive support, teacher-directed, explicit connection of concrete-representation-abstract sequencing of instruction may be needed.

3. Situates learning of target concepts and skills within authentic contexts relevant to the students’ interests and daily lives.

4. Incorporates activities to develop conceptual understanding.

5. Provides sufficient opportunity to build fluency.

6. Integrates the use of teaching tools and technology to establish greater understanding of the key idea being taught.

7. Focuses instruction to address where students are in terms of acquiring understanding, building proficiency, maintaining proficiency, generalizing, or adapting existing knowledge for new understandings.

8. Flexibly adjusts the nature of teaching and learning supports during instruction to support the learning of all students. For children receiving supplemental or intensive instruction, ongoing progress monitoring should occur to verify gains and adjust instruction as needed.

In the next section, we will discuss the importance of evaluating instructional effects and adjusting instruction when needed.

**Evaluating Instructional Effects and Adjusting Subsequent Instruction**

We have said it is not sufficient to present content and presume that most children will learn it. There must be a feedback loop to the teacher to guide the teacher's instructional efforts. This feedback must come from the student. We suggest beginning with student assessment of mastery of essential learning objectives. In Figure 1, we show the performance of a fourth grade class on a multiplication facts 0-9 assessment.
Multiplication 0-9 is a skill that students are expected to have mastered according to the CCSS by third grade and it is a skill that is prerequisite to many of the skills that students must learn at fourth grade. Hence, verifying student mastery of multiplication 0-9 is important. In the figure every bar is a child's performance in a single classroom. Scores below 40 digits correct per two minutes reflect frustration-level performance and risk for mathematics failure. Scores between 40 and 79 digits correct per two minutes reflect instructional-level performance. And scores greater than 80 digits correct per two minutes indicate mastery (and forecast the ability to retain the skill and use the skill to solve novel and more complex operations which will be required at fourth grade). In this class, not a single child has performed at the mastery level meaning most of these children will struggle with multi-digit multiplication, division, and multiplication of decimals. These students will struggle to understand division as finding an unknown factor, which is an important prerequisite to working with fractions. Hence, Figure 1 is an example of a classwide problem in mathematics.
When a classwide problem in mathematics is detected, decision teams should step back and consider whether there is a gradewide or even a schoolwide problem in mathematics. Treating systemic problems at the grade or school level (as opposed to individual student problems) is not only a more efficient solution, it is also more effective. Individual interventions that are integrated within systems that are flawed have a low probability of success. In this example, two fatal flaws would compromise the potential for success of intervening at the individual level rather than at a systems level. First, the fact that the overwhelming number of individual interventions required would reduce the resources and capacity to deliver interventions well. Second, when most children in a class are low-performing, the potential for any measurement tool to identify individual children accurately for necessary intervention will be weak and technically inadequate. In Figure 2,
the data are examined for the entire fourth grade from which the class in Figure 1 came. In Figure 2, each bar is a class’s performance on the screening measure with the y-axis indicating the percentage of students at risk by class. Teacher 1 has 67% of students at risk. In all classrooms at fourth grade, more than 50% of students are at risk of mathematics failure.

Note. Graph created on isteep and reproduced with permission from the National Center for Learning Disabilities.

In the case of a gradewide problem like this, it makes sense to look further and identify whether multiple grades at the school are experiencing similar patterns of risk. These data should cause the decision team to examine the adequacy of core instruction and taking the following corrective actions (1) verify teacher's understandings of what skills are expected across grade levels, (2) specify a calendar of instruction that paces learning effectively across the school year and across grade levels, (3) verify that excellent acquisition and
fluency-building instruction is occurring in each classroom, and (4) initiate progress monitoring measurement to verify that corrective actions improve learning outcomes over time. The school may choose to begin a classwide intervention supplement in each classroom. Importantly, whatever corrective action is chosen, there must be a feedback loop to the decision team to verify that the solution is working. Follow-up screening data are ideal for examining risk reductions across classrooms over time. In Figure 3, the percentage of students at risk by class is shown for fall and winter for each teacher. In all cases, the percentage of children at risk for mathematics failure has declined substantially with intervention indicating the intervention is a good investment of time and resources and should be continued.
Because interventions that are not managed are almost certain to fail, decision teams should tie a system of performance feedback to their intervention program (Noell et al., 1995). Performance feedback involves tracking student performance and going into classrooms with students who are lagging behind other classrooms to provide in-class coaching and support to the teacher. These coaching sessions are an important opportunity to troubleshoot instruction and ensure learning gains for students (Witt, Noell, LaFleur, & Mortenson, 1997). Without this system of support, some teachers will not be able to make the changes needed to improve student learning. The decision team must routinely examine instructional effects by classroom to know where in-class coaching and support is needed. In Figure 4, classes 1-3 are showing a need for in-class coaching and support.

Note. Figure produced in excel and reprinted with permission from the National Center for Learning Disabilities.

Mathematics performance data should also be utilized in systematic ways to intervene at the individual student level. For example, when an individual student demonstrates continued poor performance and efforts like the ones described above have not led to success then the use of particular mathematics curriculum based assessment
techniques can be utilized to pinpoint reasons for the student’s mathematics difficulties. The use of error pattern analyses, flexible mathematics interviews, and C-R-A assessments can be utilized to target what a student is able and unable to do given a particular mathematics concept or skill and what are faulty areas of mathematical thinking that impact the student’s mathematical progress (e.g., Bryant, 1996; Gersten, 1998; Ginsburg, 1987; Howell, Fox, & Morehead, 1993; Kami, 2000; Liedtke, 1988; Mercer & Mercer, 2005; Van de Walle, 2005; Zimmond, Vallecorsa, & Silverman, 1981). Allsopp, Kyger, Lovin, Gerretson, and Ray (2008) describe a process for utilizing data gathered through the integration of CRA assessment, error pattern analysis, and a flexible mathematics interview to develop an instructional hypothesis to guide interventions by pinpointing what a student can do, what they cannot do, and why. Fuchs et al. (2008) describe an assessment process by which a mathematics learning task is given to a student and then instruction or feedback is provided to help the student learn the task. The teacher records the student’s response to the instruction/feedback as a way to determine the potential for learning the given skill set. If areas of mathematical difficulties for an individual student are documented via reliable screening and curriculum based measurement processes, then the use of informal curriculum based assessment techniques can provide instructionally relevant data to guide subsequent instruction and intervention.

Necessary indicators that a teacher knows how to evaluate and adjust instruction include:

1. Utilizes periodic assessment to verify retention of learned skills.
2. Utilizes annual assessment for accountability linked to system planning and problem-solving.
3. Engages in routine monitoring of student mastery of key mathematics concepts/skills and compares classroom student data to grade level data for all students in that grade for instructional decision making. Conducts/utilizes follow-up assessment to determine when supplemental intervention has been successful and can be discontinued or when students should be transitioned to more intensive intervention procedures. Conducts/utilizes follow-up assessment conducted to verify gains for targeted foundational mathematics concepts/skills. Conducts/utilizes follow-up assessment data to verify generalized learning improvements during core instruction (e.g., verifying children perform outside of risk range on subsequent screenings and score in the proficient range on year-end accountability tests).

4. Utilizes assessment data to select instruction practices aligned with students’ levels of conceptual understanding and proficiency of mathematics learning objectives.

5. Effectively collaborates with school-based problem solving teams to describe nature of core instruction, supplemental instruction, and intensive instruction and to share informal classroom assessment data, and make appropriate instructional decisions.

Conclusion

We have attempted to describe in greater detail the evidence-based indicators included in IC Math so that State SEAs and IHEs, and other stakeholders have a substantive discussion of the characteristics of high-quality mathematics instruction. SEAs and IHEs and other stakeholders may use the IC Math to evaluate and seek to improve current state licensure requirements, state-, district-, and school-level professional development activities, and teacher preparation for PK-12 mathematics for the benefit of all students,
including students with disabilities. There are particular strengths and weaknesses to the research base related to effective mathematics instruction. We have attempted to identify those practices that have the greatest potential for affecting positive mathematics outcomes for all students, particularly struggling learners. Some practices have a stronger evidence base compared to others. The indicators and related practices described in this narrative can assist any SEA or IHE to critically evaluate their current practice, set goals for improvement, develop and implement an improvement plan, and set benchmarks for evaluating improvement.
References


